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Signatures of finite exceptional Lie algebra representations

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Abstract. The paper deals with the real exceptional Lie algebras of types E_i, $i=6, 7, 8$ and their arbitrary irreducible representations. Hermitian forms which are invariant relative to these representations are considered. Signature formulas for these forms are obtained.

1. Introduction

Let $\mathfrak{g}$ be the simple complex Lie algebra and let $\mathfrak{g}_\sigma$ be any real form of inner type for $\mathfrak{g}$. Consider an irreducible representation $\varphi\colon \mathfrak{g}\to\mathfrak{sl}(V)$. From [1] it follows that $\varphi(\mathfrak{g}_\sigma)\subset\mathfrak{su}(p, q)$, where $p+q=\dim V$. Let $\delta=p-q$. So δ is a signature, i.e. the difference between the number of positive and negative signs in the bilinear invariant in its diagonal form. Furthermore $p=\frac{1}{2}(\dim V+\delta)$ and $q=\frac{1}{2}(\dim V-\delta)$. Hence it is possible to find the number of linearly independent spacelike or timelike vectors in representation space. In [1-5] formulas for δ were given in terms of the highest weight. Lie algebras of types G_2, F_4 were considered in [2] and [4]. As follows from this paper, it is possible to obtain simple δ formulas in the case of real Lie algebras of types E_i, $i=6, 7, 8$.

The finite-dimensional representations which are used in theoretical physics are mostly low-dimensional, nevertheless the interest in general methods still grows [5].

2. Definitions

Definitions used in this paper coincide with those in [4]. Let $\mathfrak{g}_\tau$ be the fixed compact real form of the algebra $\mathfrak{g}$ and let τ be the conjugation of the algebra $\mathfrak{g}$ with respect to $\mathfrak{g}_\tau$. Consider an involution θ of the algebra $\mathfrak{g}$ such that $\theta(\mathfrak{g}_\tau)=\mathfrak{g}_\tau$. Let $\sigma=\tau\circ\theta=\theta\circ\tau$. Denote by $\mathfrak{g}_\sigma$ the real form of the algebra $\mathfrak{g}$ such that σ is a conjugation of the algebra $\mathfrak{g}$ with respect to $\mathfrak{g}_\sigma$. The real form $\mathfrak{g}_\sigma$ is called the real form of inner type if $\theta\in\mathrm{Int}(\mathfrak{g}_\tau)$. Suppose $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_\tau$ such that $\theta(\mathfrak{t})=\mathfrak{t}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t}^{\mathbb{C}}=\mathfrak{h}$, R is a root system associated with the pair $(\mathfrak{g}, \mathfrak{h})$. Let $B(\ ,\ )$ be a Killing form of $\mathfrak{g}$, and let $(\ ,\ )=(-1/(2\pi)^2)\,B(\ ,\ )$ be a positive definite scalar product on $\mathfrak{t}$. Let $\alpha\in R$; by H_α denote an element of $\mathfrak{h}$ such that $B(H_\alpha, H)=\alpha(H)$ for all $H\in\mathfrak{h}$. Define the embedding $\psi\colon R\to\mathfrak{t}$ by $\psi(\alpha)=(2\pi\sqrt{-1})\,H_\alpha$ for all $\alpha\in R$. Suppose $\Pi=\{\alpha_1, \ldots, \alpha_r\}$ is a set of the simple roots of R, $\{H_i\}_{i=1}^r$ is the basis of $\mathfrak{t}$ such that $(H_i, \alpha_j)=\delta_{ij}$, $i, j=1, \ldots, r$. If $\theta\in\mathrm{Int}(\mathfrak{g}_\tau)$, then without loss of generality,

$\theta = \exp(\mathrm{ad}(H_{i_0}/2))$ for some i_0, $1 \leqslant i_0 \leqslant r$ [6]. Let $R^\vee$ be the root system dual to R, that is

$$R^\vee = \left\{ \frac{2\alpha}{(\alpha, \alpha)} \middle| \alpha \in R \right\}.$$

Suppose W is a Weyl group of R, and $P(R^\vee)$ is a group of weights for $R^\vee$ [7], where $P(R^\vee)$ is generated by the elements $\{H_i\}_{i=1}^r$ mentioned above. Let λ be the highest weight of the representation $\varphi: \mathfrak{g} \to \mathfrak{sl}(V)$ and let χ_λ be the character of the representation φ. According to the Weyl character formula we have $A_\rho(H)\, \chi_\lambda(H) = A_{\lambda+\rho}(H)$, where

$$A_{\lambda+\rho}(H) = \sum_{S \in W} \det s \exp(2\pi\sqrt{-1}\, (s(\lambda+\rho), H))$$

and

$$\rho = \tfrac{1}{2} \sum_{\beta \in R, \beta > 0} \beta$$

is half the sum of the positive roots R.

Then [6]

$$A_\rho(H) = (2\sqrt{-1})^l \prod_{\beta \in R, \beta > 0} \sin(\pi(\beta, H)) \tag{1}$$

where l is the number of positive roots. Denote by ω_i, $i = 1, \ldots, \mathrm{rang}(\mathfrak{g})$ basis representations of the algebra $\mathfrak{g}$, that is

$$\frac{2(\omega_i, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{ik}$$

where $\alpha_k \in \Pi$, $i, k = 1, \ldots, \mathrm{rang}(\mathfrak{g})$. In accordance with [4] we shall call elements H_1 and $H_2 \in \mathfrak{h}$ equivalent if there exists $s \in W$ such that $s(H_1) - H_2 \in P(R^\vee)$ and we shall write $H_1 \equiv H_2 (\mathrm{mod}\ P(R^\vee))$.

Lemma 1 [4]. Let $\mathfrak{g}_\sigma$ be a real form of simple complex algebra $\mathfrak{g}$, $\theta = \sigma \circ \tau = \exp(\mathrm{ad}(H_{i_0}/2))$ and χ_λ be a character of the irreducible representation $\varphi: \mathfrak{g} \to \mathfrak{sl}(V)$. Then

$$|\delta| = |\chi_\lambda(H)| = \left| \lim_{t \to 1} \frac{A_{\lambda+\rho}(tH)}{A_\rho(tH)} \right| \tag{2}$$

where $H \equiv H_{i_0}/2\ (\mathrm{mod}\ P(R^\vee))$.

3. The case $\mathfrak{g} = E_6$, $\mathfrak{g}_\sigma = E\,\mathrm{II}$

The Dynkin diagram for E_6 is

```
α1 α3 α4 α5 α6
0—0—0—0—0
      |
      0  α2
```

We shall take the roots realization from [8], that is

$$\begin{aligned} &\alpha_1=\varepsilon_2-\varepsilon_3 \qquad \alpha_2=\tfrac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7-\varepsilon_8) \\ &\alpha_3=\varepsilon_3-\varepsilon_4 \qquad \alpha_4=\varepsilon_4-\varepsilon_5 \qquad \alpha_5=\varepsilon_5-\varepsilon_6 \qquad \alpha_6=\varepsilon_6-\varepsilon_7. \end{aligned} \tag{3}$$

By symbol

$$\begin{array}{ccccc} a & c & d & e & f \\ & & b & & \end{array}$$

denote the root $\beta=a\alpha_1+b\alpha_2+c\alpha_3+d\alpha_4+e\alpha_5+f\alpha_6$. Let R^+ be the set of positive roots $\beta\in R$, $\beta>0$. Then

$$\begin{aligned} R^+=\{\varepsilon_1-\varepsilon_8\}\cup\{\varepsilon_i-\varepsilon_j\,|\,2\leqslant i<j\leqslant 7\} \\ \cup\{\tfrac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4\pm\varepsilon_5\pm\varepsilon_6\pm\varepsilon_7-\varepsilon_8)\,|\,3\text{ “+” sign and }3\text{ “−” sign}\}. \end{aligned}$$

Let ω_i, $i=1,\ldots,6$ be basis representations of E_6. By symbol

$$\begin{array}{ccccc} \lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ 0 - & 0 - & 0 - & 0 - & 0 \\ & & | & & \\ & & 0\ \lambda_2 & & \end{array}$$

denote the representation with the highest weight

$$\lambda=\sum_{j=1}^{6}\lambda_j\omega_j.$$

The element $H=\frac{1}{2}H_2$ defines automorphism $\theta=\exp(\mathrm{ad}\,H)$. Then

$$H_i=\frac{2\omega}{(\alpha_i,\alpha_i)}=\omega_i \qquad i=1,\ldots,6.$$

Furthermore

$$\begin{aligned} \frac{H_2}{2} &\equiv\frac{H_2}{2}+2H_1+2H_3+2H_4+2H_5+2H_6=\tfrac{1}{2}(\rho+(3\rho-3\omega_2)) \\ &=\tfrac{1}{2}\left(\rho+3\begin{bmatrix}7 & 13 & 18 & 13 & 7 \\ & & 9 & & \end{bmatrix}\right) \\ &\equiv\tfrac{1}{2}\left(\rho+\begin{bmatrix}1 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{bmatrix}\right) \\ &\equiv\tfrac{1}{2}\left(\rho-7\begin{bmatrix}1 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{bmatrix}\right) \\ &=\tfrac{1}{2}(s_\beta(\rho))\equiv\tfrac{1}{2}\rho\ (\mathrm{mod}\ P(R^\vee)) \end{aligned}$$

where $s_\beta\in W$ is the reflection defined by the root

$$\beta=\begin{bmatrix}1 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{bmatrix}$$

that is

$$s_\beta(v)=v-\frac{2(\beta,v)}{(\beta,\beta)}\beta.$$

Hence from (2) it follows that

$$|\delta|=\left|\lim_{t\to 1}\frac{A_{\lambda+\rho}(\frac{1}{2}t\rho)}{A_\rho(\frac{1}{2}t\rho)}\right|=\left|\lim_{t\to 1}\frac{A_\rho(\frac{1}{2}t(\lambda+\rho))}{A_\rho(\frac{1}{2}t\rho)}\right|. \tag{4}$$

Since

$$\rho=\tfrac{1}{2}\sum_{\beta\in R^+}\beta=\sum_{j=1}^{\sigma}\omega_j$$

we have

$$|A_\rho(\tfrac{1}{2}t\rho)|=2^{36}\prod_{\beta\in R^+}\sin(\pi t(\beta,\rho))\simeq 2^{36}2^9 3^3 5(\pi(t-1))^{16} \tag{5}$$

where we have kept only the lowest degree terms when $t\to 1$. Suppose

$$X(\lambda)=\{\beta|\,\beta\in R^+,(\beta,\tfrac{1}{2}(\lambda+\rho))\in\mathbb{Z}\},\ C_\lambda=\prod_{\beta\in X(\lambda)}(\beta,\tfrac{1}{2}(\lambda+\rho)).$$

From (1) it follows that

$$|A_\rho(\tfrac{1}{2}t(\lambda+\rho))|\simeq 2^{36}C_\lambda(\pi(t-1))^{\mathrm{card}(X(\lambda))}\quad\text{when } t\to 1. \tag{6}$$

Hence the limit in (4) depends on the value of card($X(\lambda)$). The value of card($X(\lambda)$) depends on whether $\lambda_j, j=1,\ldots,6$ are even or odd. So it is necessary to consider $2^{\mathrm{rang}(\mathfrak{g})}$ cases to evaluate card($X(\lambda)$).

From table 1 it follows that card($X(\lambda)$) = 16 or 20 or 36. The foregoing proves the theorem.

Table 1. Card($X(\lambda)$), $\mathfrak{g}=E_6$.

Representation	λ_1 λ_3 λ_4 λ_5 λ_6 λ_2					Card($X(\lambda)$)
Set S_{11}	*a e a e a* *e*	*e o a e a* *e*	*a o e o a* *o*	*e e e o a* *o*	*o e o o a* *o*	
Set S_{12}	*a e a o e* *e*	*e o a o e* *e*	*o o a o o* *e*	*a o e e e* *o*	*a o o e o* *o*	
	e e e e e *o*	*o e e e o* *o*	*e e o e o* *o*	*o e o e e* *o*		16
Set S_{21}	*o o a e a* *e*	*e a o o a* *o*	*o e e o a* *o*	*o o o o e* *o*		
Set S_{22}	*a e a o o* *e*	*a o o e e* *o*	*a o e e o* *o*	*e o a o o* *e*	*o o a o e* *o*	
	e e e e o *o*	*o e e e e* *o*	*e e o e e* *o*	*o e o e o* *o*		20
Set S_{31}	*o o o o o* *o*					36

Symbol $e(o)$ in the column λ_i denotes an even (odd) λ_i.
Symbol a denotes any λ_i independent of whether it is even or odd.

Theorem 1. Let $\mathfrak{g}=E_6$ and let $\mathfrak{g}_\sigma=E\,\mathrm{II}$. Suppose

$$\lambda=\sum_{j=1}^{6}\lambda_j\omega_j$$

is the highest weight of arbitrary representation $\varphi: E_6\to\mathfrak{sl}(V)$,

$$X(\lambda)=\{\beta|\beta\in R^+, (\beta,\tfrac{1}{2}(\lambda+\rho))\in\mathbb{Z}\}\qquad C_\lambda=\prod_{\beta\in X(\lambda)}(\beta,\tfrac{1}{2}(\lambda+\rho)).$$

If $\mathrm{card}(X(\lambda))=16$, then

$$|\delta|=\frac{C_\lambda}{2^9 3^3 5}\tag{7}$$

if $\mathrm{card}(X(\lambda))>16$, then $\delta=0$.

4. The case $\mathfrak{g}=E_6$, $\mathfrak{g}_\sigma=E\,\mathrm{III}$

The automorphism $\theta=\exp(\mathrm{ad}\,H)$ is defined by the elements $\frac{1}{2}H_1$ or $\frac{1}{2}H_6$.

Lemma 2. Let

$$\lambda=\sum_{j=1}^{6}\lambda_j\omega_j$$

be the highest weight of the representation $\varphi: E_6\to\mathfrak{sl}(V)$ and let χ_λ be the character of this representation. Then $|\chi_\lambda(\frac{1}{2}H_1)|=|\chi_\lambda(\frac{1}{2}H_6)|=|\delta|=|\chi_\lambda(\frac{1}{2}(\rho+\omega_2+\omega_6))|$.

Proof.

$$\begin{aligned}\tfrac{1}{2}H_1&\equiv\tfrac{1}{2}H_1+\sum_{j=2}^{6}H_j=\tfrac{1}{2}\left(\rho+\omega_2+\omega_6+\begin{bmatrix}5&10&\begin{smallmatrix}14\\7\end{smallmatrix}&10&5\end{bmatrix}\right)\\&\equiv\tfrac{1}{2}\left(\rho+\omega_2+\omega_6+\begin{pmatrix}1&2&\begin{smallmatrix}2\\1\end{smallmatrix}&2&1\end{pmatrix}\right)\\&\equiv\tfrac{1}{2}(\rho+\omega_2+\omega_6)\ (\mathrm{mod}\ P(R^\vee)).\end{aligned}$$

Then

$$\begin{aligned}\tfrac{1}{2}H_6&\equiv\tfrac{1}{2}H_6+\sum_{j=1}^{6}H_j=\tfrac{1}{2}\left(\rho+\omega_2+\omega_6+\begin{pmatrix}7&13&\begin{smallmatrix}18\\9\end{smallmatrix}&13&7\end{pmatrix}\right)\\&\equiv\tfrac{1}{2}\left(\rho+\omega_2+\omega_6+\begin{pmatrix}1&1&\begin{smallmatrix}2\\1\end{smallmatrix}&1&1\end{pmatrix}\right)\\&\equiv\tfrac{1}{2}(\rho+\omega_2+\omega_6)\ (\mathrm{mod}\ P(R^\vee)).\end{aligned}$$

So

$$|\chi_\lambda(\tfrac{1}{2}H_1)|=|\chi_\lambda(\tfrac{1}{2}(\rho+\omega_2+\omega_6))|=|x_\lambda(\tfrac{1}{2}H_6)|$$

and lemma 2 is proved.

From lemma 2 it follows that

$$|\delta| = \left|\lim_{t\to 1} \frac{A_{\rho+\omega_2+\omega_6}(\frac{1}{2}t(\lambda+\rho))}{A_\rho(\frac{1}{2}t(\rho+\omega_2+\omega_6))}\right| = \left|\lim_{t\to 1} \frac{A_\rho(\frac{1}{2}t(\lambda+\rho))}{A_\rho(\frac{1}{2}t(\rho+\omega_2+\omega_6))}\chi_{\omega_2+\omega_6}(\tfrac{1}{2}t(\lambda+\rho))\right|. \tag{8}$$

Then keeping only the lowest degree terms when $t\to 1$ we find

$$|A_\rho(\tfrac{1}{2}t(\rho+\omega_2+\omega_6))| \simeq 2^{36}2^{11}3^5 5^2 7(\pi(t-1))^{20}.$$

Hence from (6) and (8) we derive

$$|\delta| = \left|\lim_{t\to 1} \frac{C_\lambda(\pi(t-1))^{\mathrm{card}(X(\lambda))}}{2^{11}3^5 5^2 7(\pi(t-1))^{20}}\chi_{\omega_2+\omega_6}(\tfrac{1}{2}t(\lambda+\rho))\right|. \tag{9}$$

Consider a representation

```
0   0   0   0   1
0———0———0———0———0
        |
        0  1
```

of the algebra E_6. By dim V denote the dimension of this representation. Then dim $V=1728$. $\eta_0=\omega_2+\omega_6=\frac{1}{6}(9,1,1,1,1,1,-5,-9)$ is the highest weight vector of this representation, where the vector is defined by its components in the basis $\varepsilon_1,\ldots,\varepsilon_8$ (3). η_0 and two other vectors $\eta_1=\frac{1}{3}(3,2,2,-1,-1,-1,-1,-3)$ and $\eta_2=\frac{1}{6}(3,1,1,1,1,1,-5,-3)$ are the dominant weights of the representation. Using the results of [8] we find all weight vectors of this representation. They are $w(\eta_i)$, where $w\in W$, $i=0,1,2$. By dim V^{η_i}, $i=0,1,2$ denote the weight multiplicities. Then dim $V^{\eta_0}=1$, dim $V^{\eta_1}=4$, dim $V^{\eta_2}=16$. We have used the coset decomposition of the Weyl group W with respect to the Weyl group $W_{\mathrm{su}(2)\times\mathrm{su}(6)}$ of a classical regular subalgebra $\mathrm{su}(2)\times\mathrm{su}(6)$ embedded naturally in E_6 [8]. Namely,

```
α1  α3  α4  α5  α6       α1  α3  α4  α5  α6
0———0———0———0———0   ⊂   0———0———0———0———0
        |                       |
        ∅  α2                   0  α2
        |
        0  α0
```

where $\alpha_0=\varepsilon_1-\varepsilon_8$ and the root α_2 is deleted. For any E_6-dominant weight vector η_i we find all A_1+A_5-dominant vectors $w(\eta_i)$ and the orders of their $W_{\mathrm{su}(2)\times\mathrm{su}(6)}$ orbits. The results are collected in table 2. From table 2 it follows that $|\chi_{\omega_2+\omega_6}(\frac{1}{2}(\lambda+\rho))|=64$ for any λ such that $\mathrm{card}(X(\lambda))=20$. In fact

$$\begin{aligned}|\chi_{\omega_2+\omega_6}(\tfrac{1}{2}(\lambda+\rho))| &= \left|\sum_{\mu i}\exp(2\pi\sqrt{-1}(\mu_i,\tfrac{1}{2}(\lambda+\rho)))\dim V^{\mu_i}\right| \\ &= \left|\sum_{\mu i}\exp(\pi\sqrt{-1}(\mu_i-\eta_0,\lambda+\rho))\dim V^{\mu_i}\right| \end{aligned} \tag{10}$$

where the sum in (10) embraces all weights μ_i from table 2. But

$$\lambda+\rho=\sum_{j=1}^{6}(\lambda_j+1)\omega_j$$

Table 2. Representation $\begin{smallmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 - & 0 - & 0 - & 0 - & 0 \\ & & \mid & & \\ & & 0 & 1 & \end{smallmatrix}$ of E_6.

	$\eta_0=\frac{1}{6}(9, 1, 1, 1, 1, 1, -5, -9)$ dim $V^{\eta_0}=1$		$\eta_1=\frac{1}{6}(3, 2, 2, -1, -1, -1, -1, -3)$, dim $V^{\eta_1}=4$	
$w(\eta)$	$\frac{1}{6}(9, 1, 1, 1, 1, 1, -5, -9)$	12	$\frac{1}{3}(3, 2, 2, -1, -1, -1, -1, -3)$	30
	$\frac{1}{3}(3, 2, 2, 2, -1, -1, -4, -3)$	120	$\frac{1}{6}(3, 7, 1, 1, 1, -5, -5, -3)$	120
	$\frac{1}{6}(3, 7, 7, 1, -5, -5, -5, -3)$	120	$\frac{1}{3}(0, 2, 2, 2, -1, -1, -4, 0)$	60
	$\frac{1}{6}(3, 7, 1, 1, 1, 1, -11, -3)$	60	$\frac{1}{3}(0, 5, -1, -1, -1, -1, -1, 0)$	6
	$\frac{1}{3}(0, 5, 2, -1, -1, -1, -4, 0)$	120		
	$\eta_2=\frac{1}{6}(3, 1, 1, 1, 1, 1, -5, -3)$, dim $V^{\eta_2}=16$			
$w(\eta)$	$\frac{1}{6}(3, 1, 1, 1, 1, 1, -5, -3)$	12	$\frac{1}{3}(0, 2, 2, -1, -1, -1, -1, 0)$	15

The vectors η_i, $i=0, 1, 2$ are E_6-dominant. The vectors $w(\eta_i)$ are A_i+A_5 dominant.

and

$$\mu_i - \eta_0 = \sum_{j=1}^{6} n_{ij}\alpha_j,$$

where $n_{ij}\in\mathbb{Z}$ and $\alpha_j\in\Pi$, $j=1, \ldots, 6$. Hence

$$\exp(\pi\sqrt{-1}(\mu_i-\eta_0, \lambda+\rho)) = \exp(\pi\sqrt{-1}\sum_{j=1}^{6} n_{ij}(\lambda_j+1)).$$

Thus the sum in (10) will not change if

$$\lambda=\sum_{j=1}^{6}\lambda_j\omega_j$$

is replaced by

$$\bar{\lambda}=\sum_{j=1}^{6}\bar{\lambda}_j\omega_j$$

where $\bar{\lambda}_j=1$ if λ_j is odd and $\bar{\lambda}_j=0$ if λ_j is even. Hence it is sufficient to consider a finite number of the representations to evaluate

$$|\chi_{\omega_2+\omega_6}(\tfrac{1}{2}(\lambda+\rho))|.$$

For any λ such that $\mathrm{card}(X(\lambda))=20$ we derive, using straightforward calculation, that

$$|\chi_{\omega_2+\omega_6}(\tfrac{1}{2}(\lambda+\rho))|=64.$$

The foregoing proves the theorem.

Theorem 2. Suppose $\mathfrak{g}=E_6$, $\mathfrak{g}_\sigma=E\,\mathrm{III}$, and

$$\lambda=\sum_{j=1}^{6}\lambda_j\omega_j$$

is the highest weight of arbitrary representation $\varphi: E_6\to\mathfrak{sl}(V)$.

If $\mathrm{card}(X(\lambda))=20$, then

$$|\delta|=\frac{C_\lambda}{2^5 3^5 5^2 7}. \tag{11}$$

If $\mathrm{card}(X(\lambda))=36$, then $\delta=0$.
If $\mathrm{card}(X(\lambda))=16$, then

$$|\delta|=\frac{C_\lambda}{2^{11}3^5 5^2 7}\frac{1}{4!}\left|\sum_{\mu_i}\cos(\pi(\mu_i-\eta_0,\lambda+\rho))\,(\mu_i,\lambda+\rho)^4\dim V^{\mu_i}\right| \tag{12}$$

where the sum in (12) embraces all weights μ_i from table 2, and $\eta_0=\frac{1}{6}(9,1,1,1,1,1,-5,-9)$ is the highest weight of the representation

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0- & 0- & 0- & 0- & 0 \\ & & | & & \\ & & 0\ 1 & & \end{array}$$

$$\lambda+\rho=\sum_{j=1}^{8}h_j\varepsilon_j$$

where

$$h_1=\tfrac{1}{6}(3\lambda_1+6\lambda_2+6\lambda_3+9\lambda_4+6\lambda_5+3\lambda_6+33)$$

$$h_2=\tfrac{1}{6}(5\lambda_1+4\lambda_3+3\lambda_4+2\lambda_5+\lambda_6+15)$$

$$h_3=\tfrac{1}{6}(-\lambda_1+4\lambda_3+3\lambda_4+2\lambda_5+\lambda_6+9)$$

$$h_4=\tfrac{1}{6}(-\lambda_1-2\lambda_3+3\lambda_4+2\lambda_5+\lambda_6+3)$$

$$h_5=\tfrac{1}{6}(-\lambda_1-2\lambda_3-3\lambda_4+2\lambda_5+\lambda_6-3)$$

$$h_6=\tfrac{1}{6}(-\lambda_1-2\lambda_3-3\lambda_4-4\lambda_5+\lambda_6-9)$$

$$h_7=\tfrac{1}{6}(-\lambda_1-2\lambda_3-3\lambda_4-4\lambda_5-5\lambda_6-15)$$

$$h_8=\tfrac{1}{6}(-3\lambda_1-6\lambda_2-6\lambda_3-9\lambda_4-6\lambda_5-3\lambda_6-33).$$

Using formulas (7), (11) and (12) it is possible to find $|\delta|$ for any representation φ. The values of $|\delta|$ for some representations are collected in table 3.

5. The case $\mathfrak{g}=E_7$

The Dynkin diagram for E_7 is

$$\begin{array}{cccccc} \alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ 0- & 0- & 0- & 0- & 0- & 0 \\ & & | & & & \\ & & 0\ \ \alpha_2 & & & \end{array}.$$

We shall take the roots realization from [8], that is

$$\begin{array}{llll} \alpha_1=\varepsilon_7-\varepsilon_8 & \alpha_2=\tfrac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5+\varepsilon_6+\varepsilon_7+\varepsilon_8) & \alpha_3=\varepsilon_6-\varepsilon_7 & \\ \alpha_4=\varepsilon_5-\varepsilon_6 & \alpha_5=\varepsilon_4-\varepsilon_5 & \alpha_6=\varepsilon_3-\varepsilon_4 & \alpha_7=\varepsilon_2-\varepsilon_3. \end{array} \tag{13}$$

Table 3. The values of $|\delta|$, $\mathfrak{g}=E_6$.

Representation λ_1 λ_3 λ_4 λ_5 λ_6 0—0—0—0—0 | 0 λ_2	dim V	$\|\delta\|$ for E II	$\|\delta\|$ for E III	Representation λ_1 λ_3 λ_4 λ_5 λ_6 0—0—0—0—0 | 0 λ_2	dim V	$\|\delta\|$ for E II	$\|\delta\|$ for E III
1 0 0 0 0 0—0—0—0—0 | 0 0	27	3	5	1 0 0 0 1 0—0—0—0—0 | 0 0	650	10	10
0 0 0 0 0 0—0—0—0—0 | 0 1	78	2	14	0 0 1 0 0 0—0—0—0—0 | 0 1	105600	0	640
0 1 0 0 0 0—0—0—0—0 | 0 0	351	9	1	1 1 0 0 0 0—0—0—0—0 | 0 0	5824	0	64
0 0 1 0 0 0—0—0—0—0 | 0 0	2925	35	45	2 0 0 0 0 0—0—0—0—0 | 0 0	351	15	31
1 0 0 0 0 0—0—0—0—0 | 0 0	1728	0	64				

By symbol

$$\begin{matrix} a & c & d & e & f & g \\ & & b & & & \end{matrix}$$

denote the root $\beta=a\alpha_1+b\alpha_2+c\alpha_3+d\alpha_4+e\alpha_5+f\alpha_6+g\alpha_7$. Suppose R^+ is a set of positive roots. Then

$$R^+=\{\varepsilon_i-\varepsilon_j \mid 1\leqslant i<j\leqslant 8\}$$
$$\cup\{\tfrac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4\pm\varepsilon_5\pm\varepsilon_6\pm\varepsilon_7\pm\varepsilon_8) \mid 3 \text{ "+" sign and } 4 \text{ "−" sign}\}.$$

By symbol

λ_1 λ_3 λ_4 λ_5 λ_6 λ_7
0—0—0—0—0—0
|
0 λ_2

denote the representation of the highest weight

$$\lambda=\sum_{j=1}^{7}\lambda_j\omega_j.$$

Let $\mathfrak{g}_\sigma$ be any real form of the algebra $\mathfrak{g}$ and let $\frac{1}{2}H_{i_0}$ generate automorphism $\theta=\exp(\mathrm{ad}\,H)$. Discussing this in the same way as previously we reduce the element $\frac{1}{2}H_{i_0}$

Table 4. The elements $\frac{1}{2}H_{i_0}(\mathrm{mod}\ P(R^\vee))$.

$\mathfrak{g}$	E_7		E_8		
$\mathfrak{g}_\sigma$	E V	E VI	E VII	E VIII	E IX
$\frac{1}{2}H_{i_0}$	$\frac{1}{2}H_2$	$\frac{1}{2}H_1$	$\frac{1}{2}H_7$	$\frac{1}{2}H_1$	$\frac{1}{2}H_8$
$\frac{1}{2}H_{i_0}(\mathrm{mod}\ P(R^\vee))$	$\frac{1}{2}\rho$	$\frac{1}{2}(\rho+\omega_2)$	$\frac{1}{2}(\rho+\omega_1+\omega_3)$	$\frac{1}{2}\rho$	$\frac{1}{2}(\rho+\omega_1+\omega_3)$

to $\frac{1}{2}H_{i_0}$ (mod $P(R^\vee)$) in the form $\frac{1}{2}(\rho+\eta_0)$. The element η_0 may or may not be zero. The results are collected in table 4.

Let $\mathfrak{g}_\sigma = E$ V. Then from (2) we derive, using table 4, that

$$|\delta| = \left|\lim_{t\to 1}\frac{A_\rho(\frac{1}{2}t(\lambda+\rho))}{A_\rho(\frac{1}{2}t\rho)}\right|. \tag{14}$$

Furthermore

$$|A_\rho(\tfrac{1}{2}t\rho)| \simeq 2^{63}2^9 3^7 5^3 7(\pi(t-1))^{28} \quad \text{when } t\to 1. \tag{15}$$

The sign "$\simeq \ldots t\to 1$" in (15) and everywhere below means that we keep only the lowest degree terms when t→1 for a function considered. Thus

$$|A_\rho(\tfrac{1}{2}t(\lambda+\rho))| \simeq 2^{63}C_\lambda(\pi(t-1))^{\mathrm{card}(X(\lambda))} \quad \text{when } t\to 1. \tag{16}$$

The limit in (14) depends on the value of card($X(\lambda)$). From table 5 it follows that card($X(\lambda)$) = 28 or 31 or 36 or 63. The foregoing proves the theorem.

Table 5. Card($X(\lambda)$), $\mathfrak{g}=E_7$.

Representation $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7\\ & & \lambda_2 & & & \end{smallmatrix}$	Card($X(\lambda)$)
$\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & e\\ & & \lambda_2 & & & \end{smallmatrix}$, where $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6\\ & & \lambda_2 & & \end{smallmatrix} \in S_{11}$ from table 1	28
$\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & o\\ & & \lambda_2 & & & \end{smallmatrix}$, where $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6\\ & & \lambda_2 & & \end{smallmatrix} \in S_{12}$ from table 1	
$\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & o\\ & & \lambda_2 & & & \end{smallmatrix}$, where $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6\\ & & \lambda_2 & & \end{smallmatrix} \in S_{11}$ from table 1	
$\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & e\\ & & \lambda_2 & & & \end{smallmatrix}$, where $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6\\ & & \lambda_2 & & \end{smallmatrix} \in S_{12}$ from table 1	
$\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & o\\ & & \lambda_2 & & & \end{smallmatrix}$, where $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6\\ & & \lambda_2 & & \end{smallmatrix} \in S_{21}$ from table 1	31
$\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & e\\ & & \lambda_2 & & & \end{smallmatrix}$, where $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6\\ & & \lambda_2 & & \end{smallmatrix} \in S_{22}$ from table 1	
$\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & e\\ & & \lambda_2 & & & \end{smallmatrix}$, where $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6\\ & & \lambda_2 & & \end{smallmatrix} \in S_{21}$ from table 1	
$\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & o\\ & & \lambda_2 & & & \end{smallmatrix}$, where $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6\\ & & \lambda_2 & & \end{smallmatrix} \in S_{22}$ from table 1	36
$\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & e\\ & & \lambda_2 & & & \end{smallmatrix}$, where $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6\\ & & \lambda_2 & & \end{smallmatrix} \in S_{31}$ from table 1	
$\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & o\\ & & \lambda_2 & & & \end{smallmatrix}$, where $\begin{smallmatrix}\lambda_1 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6\\ & & \lambda_2 & & \end{smallmatrix} \in S_{31}$ from table 1	63

Symbol $e(o)$ in the column λ_i denotes an even (odd) λ_i

Theorem 3. Let $\mathfrak{g}=E_7$ and let $\mathfrak{g}_\sigma = E\,\mathrm{V}$. Suppose

$$\lambda=\sum_{j=1}^{7}\lambda_j\omega_j$$

is the highest weight of arbitrary representation $\varphi: E_7\to \mathfrak{sl}(V)$.

If $\mathrm{card}(X(\lambda))=28$, then

$$|\delta|=\frac{C_\lambda}{2^9 3^7 5^3 7}. \tag{17}$$

If $\mathrm{card}(X(\lambda))>28$, then $\delta=0$.

Let $\mathfrak{g}_\sigma=E\,\mathrm{VI}$. Then similarly

$$|\delta|=\left|\lim_{t\to 1}\frac{A_{\rho+\omega_2}(\frac{1}{2}t(\lambda+\rho))}{A_\rho(\frac{1}{2}t(\rho+\omega_2))}\right|=\left|\lim_{t\to 1}\frac{A_\rho(\frac{1}{2}t(\lambda+\rho))}{A_\rho(\frac{1}{2}t(\rho+\omega_2))}\chi_{\omega_2}(\tfrac{1}{2}t(\lambda+\rho))\right| \tag{18}$$

$$|A_\rho(\tfrac{1}{2}t(\rho+\omega_2))|\simeq 2^{63}2^{20}3^9 5^4 7^2(\pi(t-1))^{31}\quad \text{when } t\to 1.$$

Furthermore from (18) and (16) it follows that

$$|\delta|=\left|\lim_{t\to 1}\frac{C_\rho(\pi(t-1))^{\mathrm{card}(X(\lambda))}}{2^{20}3^9 5^4 7^2(\pi(t-1))^{31}}\chi_{\omega_2}(\tfrac{1}{2}t(\lambda+\rho))\right|. \tag{19}$$

Consider a representation

```
0 0 0 0 0 0
0—0—0—0—0—0
    |
    0 1
```

of the algebra E_7. Using the results of [8] we find all weight vectors of this representation. We have used the coset decomposition of the Weyl group W with respect to the Weyl group $W_{\mathfrak{su}(8)}$ of a classical regular subalgebra $\mathfrak{su}(8)$ embedded naturally in E_7 [8]. Namely

```
α1 α3 α4 α5 α6 α7 α0     α1 α3 α4 α5 α6 α7
0—0—0—0—0—0—0       ⊂  0—0—0—0—0—0
    |                        |
    ∅  α2                     0  α2
```

where $\alpha_0=\varepsilon_1-\varepsilon_2$ and the root α_2 is deleted. For any E_7-dominant weight vector η_i we find all A_7-dominant vectors $w(\eta_i)$ and the order of their $W_{\mathfrak{su}(8)}$ orbits. The results are collected in table 6. Let $\mathrm{card}(X(\lambda))=31$. Then from table 6 it follows that $|\chi_{\omega_2}(\frac{1}{2}(\lambda+\rho))|=16$.

Table 6. Representation

```
0 0 0 0 0 0
0—0—0—0—0—0
    |
    0 1
```

of E_7.

$\eta_0=\frac{1}{4}(7,-1,-1,-1,-1,-1,-1,-1)$, dim $V^{\eta_0}=1$		$\eta_1=\frac{1}{4}(3,3,-1,-1,-1,-1,-1,-1)$, dim $V^{\eta_1}=6$	
$\frac{1}{4}(7,-1,-1,-1,-1,-1,-1,-1)$	8	$\frac{1}{4}(3,3,-1,-1,-1,-1,-1,-1)$	28
$\frac{1}{4}(1,1,1,1,1,1,1,-7)$	8	$\frac{1}{4}(1,1,1,1,1,1,-3,-3)$	28
$w(\eta)$ $\frac{1}{4}(5,1,1,1,1,-3,-3,-3)$	280		
$\frac{1}{4}(3,3,3,-1,-1,-1,-1,-5)$	280		

The foregoing proves the theorem.

Theorem 4. Suppose $\mathfrak{g}=E_7$, $\mathfrak{g}_\sigma=E\,\mathrm{VI}$ and $\lambda=\Sigma_{j=1}^{7}\lambda_j\omega_j$ is the highest weight of arbitrary representation $\varphi: E_7\to\mathfrak{sl}(V)$.

If $\mathrm{card}(X(\lambda))=31$, then

$$|\delta|=\frac{C_\lambda}{2^{16}3^9 5^4 7^2}. \tag{20}$$

If $\mathrm{card}(X(\lambda))=36$ or 63, then $\delta=0$.
If $\mathrm{card}(X(\lambda))=28$, then

$$|\delta|=\frac{C_\lambda}{2^{20}3^9 5^4 7^2}\frac{1}{3!}\left|\sum_{\mu_i}\cos(\pi(\mu_i-\eta_0,\lambda+\rho))(\mu_i,\lambda+\rho)^3\dim V^{\mu_i}\right| \tag{21}$$

where the sum in (21) embraces all weights μ_i from table 6, and $\eta_0=\frac{1}{4}(7,-1,-1,-1,-1,-1,-1,-1)$ is the highest weight of the representation

```
0  0  0  0  0  0
0—0—0—0—0—0
      |
      0 1
```

$$\lambda+\rho=\sum_{j=1}^{8}h_j\varepsilon_j$$

where

$$h_1=\tfrac{1}{4}(4\lambda_1+7\lambda_2+8\lambda_3+12\lambda_4+9\lambda_5+6\lambda_6+3\lambda_7+49)$$

$$h_2=\tfrac{1}{4}(-\lambda_2+\lambda_5+2\lambda_6+3\lambda_7+5)$$

$$h_3=\tfrac{1}{4}(-\lambda_2+\lambda_5+2\lambda_6-\lambda_7+1)$$

$$h_4=\tfrac{1}{4}(-\lambda_2+\lambda_5-2\lambda_6-\lambda_7-3)$$

$$h_5=\tfrac{1}{4}(-\lambda_2-3\lambda_5-2\lambda_6-\lambda_7-7)$$

$$h_6=\tfrac{1}{4}(-\lambda_2-4\lambda_4-3\lambda_5-2\lambda_6-\lambda_7-11)$$

$$h_7=\tfrac{1}{4}(-\lambda_2-4\lambda_3-4\lambda_4-3\lambda_5-2\lambda_6-\lambda_7-15)$$

$$h_8=\tfrac{1}{4}(-4\lambda_1-\lambda_2-4\lambda_3-4\lambda_4-3\lambda_5-2\lambda_6-\lambda_7-19).$$

Let $\mathfrak{g}_\sigma=E\,\mathrm{VII}$. Then similarly

$$|\delta|=\left|\lim_{t\to 1}\frac{C_\lambda\,(\pi(t-1))^{\mathrm{card}(X(\lambda))}}{2^{25}3^{10}5^5 7^3 11(\pi(t-1))^{36}}\chi_{\omega_1+\omega_3}(\tfrac{1}{2}t(\lambda+\rho))\right|.$$

Consider a representation

```
1  1  0  0  0  0
0—0—0—0—0—0
      |
      0 0
```

of the algebra E_7. Discussing this as in the previous cases we find all weight vectors of this representation. Furthermore if $\mathrm{card}(X(\lambda))=36$ then $|\chi_{\omega_1+\omega_3}(\frac{1}{2}(\lambda+\rho))|=4096$. The foregoing proves the theorem.

Theorem 5. Suppose $\mathfrak{g}=E_7$, $\mathfrak{g}_\sigma=E\,\mathrm{VII}$, and

$$\lambda=\sum_{j=1}^{7}\lambda_j\omega_j$$

is the highest weight of arbitrary representation $\varphi\colon E_7\to\mathfrak{sl}(V)$.

If $\mathrm{card}(X(\lambda))=36$, then

$$|\delta|=\frac{C_\lambda}{2^{13}3^{10}5^5 7^3 11}.$$

If $\mathrm{card}(X(\lambda))=31$ or 63, then $\delta=0$.
If $\mathrm{card}(X(\lambda))=28$, then

$$|\delta|=\frac{C_\lambda}{2^{25}3^{10}5^5 7^3 11}\,\frac{1}{8!}\left|\sum_{\mu_i}\cos(\pi(\mu_i-\eta_0,\lambda+\rho))(\mu_i,\lambda+\rho)^8\dim V^{\mu_i}\right| \tag{22}$$

Table 7. The values of $|\delta|$, $\mathfrak{g}=E_7$.

Representation λ_1 λ_3 λ_4 λ_5 λ_6 λ_7 0—0—0—0—0—0 | 0 λ_2	$\|\delta\|$ for E V	$\|\delta\|$ for E VI	$\|\delta\|$ for E VII	Representation λ_1 λ_3 λ_4 λ_5 λ_6 λ_7 0—0—0—0—0—0 | 0 λ_2	$\|\delta\|$ for E V	$\|\delta\|$ for E VI	$\|\delta\|$ for E VII
1 0 0 0 0 0 0—0—0—0—0—0 | 0 0 dim $V=133$	7	5	25	0 0 0 0 0 1 0—0—0—0—0—0 | 0 0 dim $V=56$	0	8	0
0 1 0 0 0 0 0—0—0—0—0—0 | 0 0 dim $V=8645$	35	59	221	0 0 0 0 0 0 0—0—0—0—0—0 | 0 1 dim $V=912$	0	16	0
0 0 1 0 0 0 0—0—0—0—0—0 | 0 0 dim $V=365750$	350	330	350	2 0 0 0 0 0 0—0—0—0—0—0 | 0 0 dim $V=7371$	63	75	351
0 0 0 1 0 0 0—0—0—0—0—0 | 0 0 dim $V=27664$	0	144	0	0 0 0 0 0 2 0—0—0—0—0—0 | 0 0 dim $V=1463$	21	55	53
0 0 0 0 1 0 0—0—0—0—0—0 | 0 0 dim $V=1539$	27	3	27				

where the sum in (22) embraces all weight vectors μ_i of the representation

```
1  1  0  0  0  0
0—0—0—0—0—0
      |
      0 0
```

$$\eta_0 = (3, 0, 0, 0, 0, 0, -1, -2)$$

$$\lambda + \rho = \sum_{j=1}^{8} h_j \varepsilon_j$$

the elements h_i, $i = 1, \ldots, 8$ have the same meaning as in theorem 4. The values of $|\delta|$ for some representations are collected in table 7.

6. The case $\mathfrak{g} = E_8$

The Dynkin diagram for E_8 is

```
α1 α3 α4 α5 α6 α7 α8
0—0—0—0—0—0—0
      |                .
      0  α2
```

We shall take the roots realization from [8], that is

$$\alpha_1 = \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$$

$$\alpha_2 = \varepsilon_7 + \varepsilon_8 \qquad \alpha_3 = \varepsilon_7 - \varepsilon_8 \qquad \alpha_4 = \varepsilon_6 - \varepsilon_7 \qquad \alpha_5 = \varepsilon_5 - \varepsilon_6 \tag{23}$$

$$\alpha_6 = \varepsilon_4 - \varepsilon_5 \qquad \alpha_7 = \varepsilon_3 - \varepsilon_4 \qquad \alpha_8 = \varepsilon_2 - \varepsilon_3 .$$

By symbol

```
a c d e f g h
    b
```

denote the root $\beta = a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6 + g\alpha_7 + h\alpha_8$. Suppose R^+ is a set of positive roots. Then

$$R^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant 8\}$$

$$\cup \{\tfrac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6 \pm \varepsilon_7 \pm \varepsilon_8) \mid \text{even numb. of "}-\text{" signs}\}.$$

By symbol

```
λ1 λ3 λ4 λ5 λ6 λ7 λ8
0—0—0—0—0—0—0
      |
      0  λ2
```

denote the representation with the highest weight

$$\lambda = \sum_{j=1}^{8} \lambda_j \omega_j .$$

Let $\mathfrak{g}_\sigma = E$ VIII. Then from (2) we find, using table 4, that

$$|\delta| = \left| \lim_{t \to 1} \frac{A_\rho(\tfrac{1}{2}t(\lambda + \rho))}{A_\rho(\tfrac{1}{2}t\rho)} \right| .$$

Table 8. Card$(X(\lambda))$, $\mathfrak{g}=E_8$.

Representation $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&\lambda_7\\&&\lambda_2&&&\end{matrix}$	Card$(X(\lambda))$
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&e&a\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{11}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&o&a\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{12}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&o&e\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{11}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&e&e\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{12}$ from table 1	56
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&o&o\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{21}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&e&o\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{22}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&o&o\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{11}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&e&o\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{12}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&o&e\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{21}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&e&e\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{22}$ from table 1	64
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&e&a\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{21}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&o&a\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{22}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&e&a\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{31}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&o&e\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{31}$ from table 1	
$\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6&o&o\\&&\lambda_2&&&&\end{matrix}$, where $\begin{matrix}\lambda_1&\lambda_3&\lambda_4&\lambda_5&\lambda_6\\&&\lambda_2&&\end{matrix}\in S_{31}$ from table 1	120

Symbols $e(o, a)$ have the same meaning as in tables 1, 5, 8.

Furthermore

$$|A_\rho(\tfrac{1}{2}t\rho)| \simeq 2^{120}2^{41}3^{19}5^8 7^5 11^2 13(\pi(t-1))^{56}$$

$$|A_\rho(\tfrac{1}{2}t(\lambda+\rho))| \simeq 2^{120}C_\rho(\pi(t-1))^{\mathrm{card}(X((\lambda))} \quad \text{when} \quad t\to 1.$$

The foregoing proves the theorem.

Theorem 6. Suppose $\mathfrak{g}=E_8$, $\mathfrak{g}_\sigma=E\,\mathrm{VIII}$, and

$$\lambda=\sum_{j=1}^{8}\lambda_j\omega_j$$

is the highest weight of arbitrary representation $\varphi: E\to\mathfrak{sl}(V)$.
If card$(X(\lambda))=56$, then

$$|\delta|=\frac{C_\lambda}{2^{41}3^{19}5^8 7^5 11^2 13}.$$

If card$(X(\lambda))>56$, then $\delta=0$, where card$(X(\lambda))$ must be taken from table 8.

Let $\mathfrak{g}_\sigma = E\,\mathrm{IX}$. Then similarly

$$|\delta| = \left| \lim_{t\to 1} \frac{C_\lambda(\pi(t-1))^{\mathrm{card}(X(\lambda))}}{2^{50}3^{22}5^{10}7^{6}11^{3}13^{2}17(\pi(t-1))^{64}} \chi_{\omega_1+\omega_3}(\tfrac{1}{2}t(\lambda+\rho)) \right|.$$

Consider a representation

```
 1   1   0   0   0   0 0
0—0—0—0—0—0—0
        |
        0 0
```

of the algebra E_8. Discussing this as in the previous cases we find all weight vectors of this representation. Furthermore if $\mathrm{card}(X(\lambda)) = 64$, then $|\chi_{\omega_1+\omega_3}(\frac{1}{2}(\lambda+\rho))| = 2^{14}$. The foregoing proves the theorem.

Theorem 7. Suppose $\mathfrak{g} = E_8$, $\mathfrak{g}_\sigma = E\,\mathrm{IX}$, and

$$\lambda = \sum_{j=1}^{8} \lambda_j \omega_j$$

is a highest weight of arbitrary representation $\varphi: E_8 \to \mathfrak{sl}(V)$.

If $\mathrm{card}(X(\lambda)) = 64$, then

$$|\delta| = \frac{C_\lambda}{2^{36}3^{22}5^{10}7^{6}11^{3}13^{2}17}.$$

If $\mathrm{card}(X(\lambda)) = 120$, then $\delta = 0$.
If $\mathrm{card}(X(\lambda)) = 56$, then

$$|\delta| = \frac{C_\lambda}{2^{50}3^{22}5^{10}7^{6}11^{3}13^{2}17} \frac{1}{8!} \left| \sum_{\mu_i} \cos(\pi(\mu_i - \eta_0, \lambda+\rho))(\mu_i, \lambda+\rho)^8 \dim V^{\mu_i} \right|$$

where the sum embraces all weight vectors μ_i of the representation

```
 1   1   0   0   0   0 0
0—0—0—0—0—0— 0
        |
        0 0
```

$$\eta_0 = \tfrac{1}{2}(11, 1, 1, 1, 1, 1, 1, -1) \qquad \lambda+\rho = \sum_{j=1}^{8} h_j \varepsilon_j$$

$$h_1 = \tfrac{1}{2}(4\lambda_1 + 5\lambda_2 + 7\lambda_3 + 10\lambda_4 + 8\lambda_5 + 6\lambda_6 + 4\lambda_7 + 2\lambda_8 + 46)$$

$$h_2 = \tfrac{1}{2}(\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + 2\lambda_6 + 2\lambda_7 + 2\lambda_8 + 12)$$

$$h_3 = \tfrac{1}{2}(\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + 2\lambda_6 + 2\lambda_7 + 10)$$

$$h_4 = \tfrac{1}{2}(\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + 2\lambda_6 + 8) \qquad h_5 = \tfrac{1}{2}(\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + 6)$$

$$h_6 = \tfrac{1}{2}(\lambda_2 + \lambda_3 + 2\lambda_4 + 4) \qquad h_7 = \tfrac{1}{2}(\lambda_2 + \lambda_3 + 2) \qquad h_8 = \tfrac{1}{2}(\lambda_2 - \lambda_3).$$

Table 9. The values of $|\delta|$, $\mathfrak{g}=E_8$.

<table>
<tr><th>Representation
λ_1 λ_3 λ_4 λ_5 λ_6 λ_7 λ_8
0—0—0—0—0—0—0
|
0 λ_2</th><th>$|\delta|$ for E VIII</th><th>$|\delta|$ for E IX</th><th>Representation
λ_1 λ_3 λ_4 λ_5 λ_6 λ_7 λ_8
0—0—0—0—0—0—0
|
0 λ_2</th><th>$|\delta|$ for E VIII</th><th>$|\delta|$ for E IX</th></tr>
<tr><td>0 0 0 0 0 0 1
0—0—0—0—0—0—0
|
0 0
dim $V=248$</td><td>8</td><td>24</td><td>1 0 0 0 0 0 0
0—0—0—0—0—0—0
|
0 0
dim $V=3875$</td><td>35</td><td>3</td></tr>
<tr><td>0 0 1 0 0 0 0
0—0—0—0—0—0—0
|
0 0
dim $V=6899079264$</td><td>41888</td><td>12320</td><td>0 1 0 0 0 0 0
0—0—0—0—0—0—0
|
0 0
dim $V=6696000$</td><td>960</td><td>2496</td></tr>
<tr><td>0 0 0 1 0 0 0
0—0—0—0—0—0—0
|
0 0
dim $V=146325270$</td><td>3094</td><td>17290</td><td>0 0 0 0 0 1 0
0—0—0—0—0—0—0
|
0 0
dim $V=30380$</td><td>84</td><td>140</td></tr>
<tr><td>0 0 0 0 1 0 0
0—0—0—0—0—0—0
|
0 0
dim $V=2450240$</td><td>832</td><td>1216</td><td>0 0 0 0 0 0 0
0—0—0—0—0—0—0
|
0 1
dim $V=147250$</td><td>50</td><td>494</td></tr>
</table>

The values of $|\delta|$ for some representations are collected in table 9.

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